PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 11 Test 1 2022

MARKING KEY

Task type: **Response**

Time allowed for this task: **40 mins**

Number of questions: **5**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **32 marks**

Task weighting: **10 %**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 **(6 marks)**

Write down the contrapositive statement and state whether each one is true or false.

(a) If $x \geq -3$, then $x^2 \geq 9$. (2 marks)

Solution
If $x^2 < 9$, then $x < -3$. This is false. E.g. x = - 10
Marking key/Mathematical behaviours
✓ Writes contrapositive ✓ States it is false

(b) If a quadrilateral has four right angles, then it is a square. (2 marks)

Solution
If a quadrilateral is not a square, then it does not have four right angles. This is false as the original statement is false.
Marking key/Mathematical behaviours
✓ Writes contrapositive ✓ States it is false

(c) If two rectangles are not congruent then they do not have same area. (2 marks)

Solution
If two rectangles have same area, then they are congruent. This is false as the original statement is false. Example – 6 by 4 and 8 by 3.
Marking key/Mathematical behaviours
✓ Writes contrapositive ✓ States it is false

Question 2 **(7 marks)**

(a) Use proof by contraposition to prove that if n^2 is even, then n is even, where $n \in \mathbb{Z}$. (4)

Solution
Contrapositive : If n is odd, then n^2 is odd, where $n \in \mathbb{Z}$. Proof : Since n is odd, $n = 2m + 1, m \in \mathbb{Z}$. $\therefore n^2 = (2m+1)^2 = 4m^2 + 4m + 1$ $= 2(2m^2 + 2m) + 1 = 2k + 1, where\ k \in \mathbb{Z}$ which is odd. Therefore, contrapositive is true. This implies that original statement is true, since contrapositive is true.
Marking key/Mathematical behaviours
✓ Writes contrapositive ✓ Expresses n as 2m+1 (an odd number) and obtains expression for n^2 ✓ Concludes that the contrapositive statement is true. ✓ Deduces that original statement is true.

(b) Hence or otherwise prove that, if x and y are integers and if $x^2 + y^2$ is even, then $x + y$ is even. (3)

Solution
Assume that $x, y \in \mathbb{Z}\ and\ x^2 + y^2$ is even. Then $x^2 + y^2 = 2k\ for\ k \in \mathbb{Z}$ Now $(x+y)^2 = x^2 + y^2 + 2xy = 2k + 2xy = 2(k + xy)$ Which is even, since $k + xy \in \mathbb{Z}$ Since $(x+y)^2$ is even, it follows from (a) that $x + y$ is even.
Marking key/Mathematical behaviours
✓ Expresses as $x^2 + y^2$ as even number ✓ Uses $(x+y)^2$ or similar statement ✓ Proves statement

Question 3 **(3 marks)**

A set of real numbers is given by $\{\pi,\ \sqrt{5},\ 0.\overline{36},\ \sqrt[3]{10}\}$. Identify the rational number and clearly show that it satisfies the definition of a rational number.

Solution
Let $x = 0.\overline{36}$ $100x = 36.\overline{36}$ $99x = 36$ $x = \frac{36}{99} = \frac{4}{11}$
Marking key/Mathematical behaviours
✓ Chooses $0.\overline{36}$ ✓ Rewrites equation without the recurring decimal ✓ Expresses as a fraction.

Question 4 **(8 marks)**

(a) Prove algebraically that if you add the squares of three consecutive numbers and then subtract 2, you always get a multiple of three. (4)

Solution
Let the three consecutive numbers be $x,\ x+1\ and\ x+2$.
$x^2+(x+1)^2+(x+2)^2-2=3x^2+6x+5-2$
$=3x^2+6x+3=3(x^2+2x+1)$
$=3(x+1)^2, which\ is\ a\ multiple\ of\ 3$
Marking key/Mathematical behaviours
✓ Chooses $three$ consecutive numbers correctly ✓ Expands the sum of 3 consecutive numbers and subtract 2 ✓ Simplify the above expression with a factor of 3 ✓ Makes conclusion

(b) Prove that one more than $(n+1)^2-(n-1)^2$ is always odd, where n is a positive integer. (4)

Solution
$(n+1)^2-(n-1)^2=n^2+2n+1-n^2+2n-1$
$=4n$
$\therefore\ one\ more\ is\ 4n+1$
$4n\ is\ even, since\ it\ has\ a\ factor\ of\ 2, therefore, 4n+1\ is\ odd$
Marking key/Mathematical behaviours
✓ Correctly expands expression ✓ Correctly simplifies ✓ Correctly adds one to get $4n+1$ ✓ Correctly states 4n has a factor of 2 and hence even.

Question 5 **(8 marks)**

(a) Prove that if $\boldsymbol{n^2}$ is divisible by 3, then $\boldsymbol{n}$ is divisible by 3. (4)

(Hint: Prove the contrapositive by considering two cases, when $\boldsymbol{n = 3k + 1}$ and $\boldsymbol{n = 3k + 2}$.)

Solution
$If\ n\ is\ not\ divisible\ by\ 3\ then\ n = 3k + 1\ or\ n = 3k + 2$
$Case\ 1: If\ n = 3k + 1\ then,$
$n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$
Which is not divisible by 3
$Case\ 2: If\ n = 3k + 2\ then,$
$n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$
Which is not divisible by 3
$Hence,$ if n^2 is divisible by 3, then n is divisible by 3.
Marking key/Mathematical behaviours
✓ Uses $n = 3k + 1$, when n is not divisible by 3
✓ Expands $n^2 = (3k + 1)^2$ and shows it is not divisible by 3
✓ Expands $n^2 = (3k + 2)^2$ and shows it is not divisible by 3
✓ Makes conclusion

(b) Hence, prove that $\sqrt{\mathbf{3}}$ is irrational. (4)

Solution
$Assume\ \sqrt{3}\ is\ rational,$ i.e. $\exists\ p, q \in \mathbb{Z}\ such\ that\ \sqrt{3} = \frac{p}{q},$ $where\ q \neq 0\ and\ p, q\ have\ no\ common\ factors$ Squaring both sides $3 = \frac{p^2}{q^2}$ $\Longrightarrow\ p^2 = 3q^2 \dashrightarrow eq\ 1$ $\Longrightarrow\ p^2\ is\ divisible\ by\ 3$ $\Longrightarrow\ p\ is\ divisible\ by\ 3, using\ (a)$ $\Longrightarrow\ p = 3k$ $\Longrightarrow\ p^2 = 9k^2 =\ 3q^2$ using eq 1 $3k^2 =\ q^2$ $\Longrightarrow\ q^2\ is\ divisible\ by\ 3$ $\Longrightarrow\ q\ is\ divisible\ by\ 3, using\ (a)$ But this implies that both p and q have common factor of 3 which is contradiction. Hence $\sqrt{3}\ is\ irrational,$
Marking key/Mathematical behaviours
✓ Assumes negation of given statement ✓ Expresses as a fraction and rearranges ✓ Concludes that both p and q have a common factor ✓ States original assumption is incorrect